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Market Power, Survival and Accuracy of Predictions in Financial Markets

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MARKET POWER, SURVIVAL AND ACCURACY OF PREDICTIONS IN FINANCIAL MARKETS

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Abstract. This paper aims to show that the market selection hypothesis in finance is not solely driven by the competitiveness of such markets, as was originally claimed by Alchian [1] and Friedman [4]. Within a standard intertemporal General Equilibrium framework, we allow for an agent to have enough influence on financial markets to strategically affect prices of assets traded. We then show that, as in Sandroni [15], the agent' long-run consumption will vanish if she makes less accurate predictions than the market, and maintain her market power otherwise. We conclude that the Darwinian justification to this market selection is not the only explanation for the eventual domination of agents making the most accurate predictions. Rather, we claim that the origin of market selection, and in turn of the common prior assumption in asset pricing, is associated with the ability to foresee accurately market uncertainty.

1. Introduction

Consider economic agents who trade financial assets under uncertainty. The long-standing *market selection hypothesis*, developed by Alchian [1] and Friedman [4], states that agents who do not make accurate predictions about the economic uncertainty are eventually driven out of the market. Part of their explanation for this phenomena stems from the competitiveness of financial markets; i.e., fierce competition among agents leads to the "natural selection" of those having the best forecasts.

Even though recent papers have cast a doubt on this approach (see Blume and Easley [2], Schleiffer and Summer [16] and Palomino [14] for instance), Sandroni in [15] has given a theoretical framework for which the above approach holds true.

Sandroni describes a model of dynamically complete markets where agents trade securities competitively, and where investment-consumption decisions are made endogenously. Sandroni then shows that the most prosperous agents are the ones who make the most accurate predictions, and that the agents who do not make accurate predictions are necessarily driven out of the market in the long-run.

A natural question that arises from Sandroni's work is to understand to which extend his results rely on the competitive nature of the market. In other words, we address the following question:

I would like to thank A. Gerber, T. Hens, A. Sandroni and J. Werner for interesting and stimulating conversations. All opinions and remaining errors in this paper are, beyond question, mine.

will an agent with market power always survive in a financial market, even if it makes less accurate predictions than other agents in the market?

The current paper advocates the idea that market selection is not entirely associated with perfect competition. We show that Sandroni's results still hold when one agent has enough influence on financial markets to strategically affect prices of assets traded. That is, we consider an agent (the *strategic agent*) aware that her large trading volume can affect market behavior, and making investment decisions accordingly. Typical examples are hedge funds whose trades have an impact on prices,[1] and central banks influencing interest rates. We develop an intertemporal model under uncertainty, close to Sandroni's, where one agent endogenizes the impact of her trades on an arbitrary number of competitive agents. Those agents are aggregated into a representative agent (the *market*), forming expectations about the economic uncertainty. We then show that the agent with market power will vanish in the long-run if it makes less accurate predictions than the market, and survives otherwise.[2]

The above result has both theoretical and practical implications. From a theoretical standpoint, our result legitimates the use of the common prior assumption in financial models with market power. Examples of this kind can be found in Hart [6], Hellwig [7], Hens et al. [8] and Kyle [10] among many others. The rationale for this common prior assumption is the same as for perfect competition: since agents capable of making the most accurate predictions are the only ones to survive, it is preferable to focus on them in a long-run analysis and disregard the others.

From a practical standpoint, the most natural implication of our result is for the survival of large trading companies, such a large hedge funds or even a central bank influencing asset prices through open market operations. It is suggested here, for instance, that failure to accurately assess economic uncertainty can lead a central bank to reduce domestic consumption.

At a more abstract level, our result also questions the commonly accepted view that market power is a source of persistent loss of efficiency. Even though efficiency loss will occur in the presence of such an economic entity, we defend the idea that government interventionism is not always needed to regulate this situation. Long-run disappearance of agents with market power, hence long-run lack of persistence of inefficiency, is proved here to possibly follow from incorrect strategic choices as a natural outcome of market forces.

Through our results, we advocate the idea that perfect competition is a sufficient condition for market selection, but is not necessary. We show that the Darwinian justification to this market selection, developed mostly by Alchian, is not the only explanation for the eventual domination of

[1] See for instance Hens et al. [8] for an analysis of this issue and related literature.

[2] In a sense defined later. The same concepts of accuracy of beliefs and survival as in Sandroni [15] are used here.

agents making the most accurate predictions. Thus, we argue that the origin of market selection, and in turn of the common prior assumption in asset pricing, is not associated with the market structure of financial markets. Even though we defend the idea that natural selection operates in such markets, this paper shows that other factors than competition may be critical to understanding this phenomena. In this respect, the origins of natural selection in financial markets, as described in Alchian [1] and Friedman [4], are to be reconsidered. Still, our result reinforces the idea that accuracy of predictions remains a critical part of assets trading under uncertainty.

We next informally describe the results. We consider an environment where time is discrete and continues forever. There is an arbitrary number of agents who live forever, one of them being strategic in the sense described above, while the others are usual price-takers. We assume that the price-takers are aggregated into a single *representative agent* (we do not develop how it is possible to do so, and take it as given).

In every period, there is a single consumption good available to the agents. A state of nature is also drawn from a finite set according to a given probability distribution. No assumption is made about the probability distribution. The state of nature drawn determines the level of endowments of the consumption good for each agent. We assume that endowments of consumption good are uniformly bounded away from zero and infinity.

Every agent is assumed to have subjective beliefs about the true probability of the states of nature, and to maximize the expected discounted sum of one-period utility that she gets from consuming the good.

The agents trade in a particular way. After every history, a new market for consumption good and securities opens. Securities traded are Arrow securities; i.e., securities that payoff one unit of consumption good if a particular state occurs and zero otherwise. We assume that, after every finite history, there exists at least one Arrow security with strictly positive payoff. This last assumption implies that markets are complete for every strictly positive assets prices.

The consumption-investment decisions of the strategic agent are constrained to the fact that 1) her own consumption must be affordable, and 2) that market clearing conditions on every market are endogenized. The second part of the problem faced by the strategic agent captures the market manipulation by anticipating the price formation. The consumption-investment decisions of the representative agent are simply constrained to the fact that her own consumption must be affordable.

In order to prove existence of such trades, we show that the above framework is equivalent to a new one where all future contingent decisions are made in the first period in an Arrow-Debreu fashion. We must make some restrictive assumptions on the aggregate demand of the market to prove existence (those assumptions are described in Section 4), although most of the standard aggregate demands in Economics satisfy those assumptions.

We next study the relation between agents' beliefs and their long-run consumptions. To do so, we use some concepts of accuracy of beliefs described next. Say that an arbitrary agent *makes accurate predictions* if, given enough data, the difference between the agent's beliefs and the true probabilities of the states of nature becomes arbitrary small in the long-run.

An agent *makes next period accurate predictions* if, given enough data, the difference between the agent's beliefs in the next period and the true probabilities of the states of nature becomes arbitrary small in the long-run.[3] (we provide an example in Section 5 showing that the two above concepts are distinct, although making accurate predictions implies making next-period predictions.)

We also use the notion of *entropy of beliefs*, defined to be the true expected value of the logarithm of the ratio of the agent's beliefs and the true probabilities of the states of natures in the next period. The *entropy* of an agent is then defined to be the logarithm of this agent's intertemporal discount factor plus the average entropy of her beliefs.[4]

We restrict even further the set of market aggregate demands to carry out the remainder of the analysis. Even though those restrictions seem strong, most of the aggregate demand functions used in policy recommendations satisfy them (again, these assumptions are presented in Section 4).

With the above market structure, we analyze the long-run performances of the strategic agent. Say that an arbitrary agent is *driven out of the market* when the agent's consumption converges to zero in the long-run.[5] Say also that an agent survives on the market when she is not driven out of the market.

In the above environment, all the results in Sandroni [15] extend to the strategic agent. When all the agents have the same intertemporal discount factor, we show that if the strategic agent makes accurate predictions then she will survive on the market; however there are cases where she is driven out of the market.

For instance, we show that, again when all the agents have the same intertemporal discount factor, if the representative agent makes accurate predictions and the strategic agent does not, then the strategic agent is driven out of the market. Therefore, even though making accurate prediction is a sufficient condition for survival for the strategic agent, making next period accurate predictions is not.

[3]We also say that an agent always makes inaccurate predictions if, in any period, her beliefs are uniformly bounded away from the true probabilities of the states of natures.

[4]Sandroni in [15] shows that the entropy of an agent is always negative. It is equal to zero if and only if the agent makes next-period accurate predictions.

[5]This last definition departs from Sandroni [15], where long-run accumulation of wealth is studied.

When the entropy of the representative agent is strictly smaller than the entropy of the strategic agent, we show that the strategic agent is again driven out of the market under some technical restrictions on the beliefs of the agents.

The paper is organized as follows: in Section 2 we describe the model and we make explicit an equilibrium concept with strategic trading; Section 3 is devoted to defining basic concepts and properties of the agents' beliefs; in Section 4 we show equivalence with a new framework and prove existence; Section 5 gives all the results related to the long-run equilibrium performances of the strategic agent. Finally, the technical proofs left aside earlier are given in the Appendix.

2. The model

In this section, a formal description of the model is given. The following notations are introduced in Sandroni [15].

Time is discrete and continues forever. In every period $t \in N_+$, a state of nature is drawn by nature from a set $S = \{1, ..., L\}$, where L is strictly greater than 1. Before defining how nature draws the states, we first need to introduce some notations.

Denote by S^t $(t \in N \cup \{\infty\})$ the $t-$Cartesian product of S. For every history $s_t \in S^t$ $(t \in N)$, a cylinder with base on s_t is defined to be the set $C(s_t) = \{s \in S^\infty |\ s = (s_t, ...)\}$ of all infinite histories whose t initial elements coincide with s_t. Define the set Γ_t $(t \in N)$ to be the $\sigma-$algebra which consists of all finite unions of cylinders with base on S^t.[6] The sequence $(\Gamma_t)_{t\in N}$ generates a filtration, and define Γ to be the $\sigma-$algebra generated by $\underset{t\in N}{\cup} \Gamma_t$.

Given an arbitrary probability measure Q on (S^∞, Γ), let dQ_t be the Γ_t-measurable function defined for every $s_t \in S^t$ $(t \in N)$ as

$$dQ_t(s) = Q(C(s_t)) \text{ and } dQ_0 = 1$$

where $s = (s_t, ...)$.

Given data up to and at period $t-1$ $(t \in N)$, the probability according to Q of a state of nature at period t, denoted by Q_t, is

$$Q_t(s) = \frac{dQ_t(s)}{dQ_{t-1}(s)},$$

for every $s \in S^\infty$.[7]

The posterior probability of Q given a finite history $s_t \in S^t$ $(t \in N)$, denoted by Q_{s_t}, is

$$Q_{s_t} = \frac{Q(A_{s_t})}{Q(C(s_t))} \text{ for all } A \in \Gamma,$$

[6]The set Γ_0 is defined to be the trivial $\sigma-$algebra, and $\Gamma_{-1} = \Gamma_0$.

[7]If $dQ_{t-1}(s)$=0 then $Q_t(s)$ is defined arbitrarily.

where A_{s_t} is the set of all paths $s \in S^\infty$ such that $s = (s_t, s')$ and $s' \in A$.[8]

The operators E^Q and $E^Q(.|\Gamma_t)(s)$, for every $s = (s_t, ...)$, are the expectation operators associated with Q and Q_{s_t} respectively.

In every period and after every finite history, nature draws a state of nature according to an arbitrary probability distribution P on (S^∞, Γ), with the property that $P_t(s) > 0$ for every $s \in S^\infty$ and t.

To conclude this section, we finally say that a finite history $s_{t+p} \in S^{t+p}$ *follows* a finite history $s_t \in S^t$ $(t, p \in N)$, denoted by $s_{t+p} \hookrightarrow s_t$, if there exists $s \in S^p$ such that $s_{t+p} = (s_t, s)$.

2.1. **The agents.** In this section, agents in the economy are described.

There is an infinitely-lived agent, the strategic agent, and a market composed of infinitely-lived price-takers behaving competitively. The strategic agent is denoted by M for manipulator, while all the price-takers are aggregated into a single representative agent, denoted by R for representative.

There is a single consumption good available in every period t $(t \in N_+)$. Denote by $c^i_{s_t}$ the consumption of an agent i $(i = M, R)$ in period t, after the history $s_t \in S^t$ $(t \in N_+)$.

In every period t $(t \in N_+)$ and after every history $s_t \in S^t$, each agent i $(i = M, R)$ is endowed with $w^i_{s_t}$ units of consumption goods before the state of nature is revealed to them. We assume that there exist two strictly positive constants A and B such that

$$A < w^i_{s_t} < B,$$

for every s_t and every i. This last assumption ensures that any agent can secure a strictly positive consumption along any path $s \in S^\infty$ by simply staying in autarchy, and cannot introduce an infinite amount of good in the economy.

The aggregate endowment w_{s_t}, after every history s_t $(s_t \in S^t$ and $t \in N_+)$, is given by

$$w_{s_t} = \sum_{i=M,R} w^i_{s_t}.$$

In every period $t \in N$, and after the realization of the history $s_t \in S^t$, the agents trade L one-period securities. Every security $d^j_{s_t}$ $(1 \leq j \leq L$ and $s_t \in S^t)$ purchased after the history s_t payoffs $d^j_{s_t}(l)$ unit of consumption good in period $t+1$ if state l is drawn, where $d^j_{s_t}(l) = 1$ if $j = l$ and 0 otherwise.[9] Let d_{s_t} denote the vector $(d^1_{s_t}, ..., d^L_{s_t})$ for every s_t. For sake of simplicity, the supply of each security is assumed to be 0 after every history.

[8]If $Q(C(s_t)) = 0$, then Q_{s_t} is defined to be an arbitrary probability measure on Γ.

[9]Such securities are commonly known as Arrow securities.

The price after history s_t of security $d^j_{s_t}$ $(1 \leq j \leq L)$ is denoted by $q^j_{s_t}$, for every $s_t \in S^t$ and $t \in N_+$. Let q_{s_t} denote the vector $(q^1_{s_t}, ..., q^L_{s_t})$ for every history s_t. It follows that, for every strictly positive vector of prices, markets are *complete.*

A *portfolio* θ^i for every agent i is a vector $(\theta_{s_t})_{s_t \in S^t, t \in N_+}$ of quantities held of the J securities after every history s_t, where $\theta_{s_t} = (\theta^j_{s_t})_j$ and $\theta^j_{s_t}$ is the holding of security $d^j_{s_t}$ after history s_t. The set of all portfolios for any agent i is denoted by Θ^i. Every agent i has no initial portfolio at date 0.

Every agent i $(i = M, R)$ does not have any information about P, the true probability measure from nature draws the states; however agent i has a subjective belief about nature represented by a probability measure P^i on (S^∞, Γ). Also, assume that $P^i_t(s) > 0$ for every $s \in S^\infty$ and t and i.

Every agent derives some utility after any history from consuming the only consumption good present in the economy. We assume that agent i $(i = M, R)$ ranks all the possible future consumption sequences $c = (c_{s_t})_{s_t \in S^t, t \in N_+}$ according to the utility function

$$U^i_0(c) = E^{P^i}\left(\sum_{p \in N_+} (\beta^i)^{t+p} u^i(c_{t+p})\right),$$

where $\beta^i \in (0,1)$ is the intertemporal discount factor, and u^i is a strictly increasing, strictly concave, twice-continuously differentiable function. We assume that u^i satisfies the Inada condition, namely $(u^i)'(c) \mapsto \infty$ as $c \mapsto 0$. Further restrictions on u^R are described later.

The budget constraints faced after every history s_t by agent i $(i = M, R)$ are

$$c_{s_t} + q_{s_t}\theta_{s_t} \leq w^i_{s_t} + d_{s_{t-1}}\theta_{s_{t-1}} \tag{2.1}$$

$$c_{s_t} \geq 0 \text{ and } \theta \in \Theta^i, \tag{2.2}$$

where $s_t \hookrightarrow s_{t-1}$ and with the convention that $\theta_{s_{-1}} = d_{s_{-1}} = \underbrace{(0, ..., 0)}_{L \text{ times}}$.

For every i $(i = M, R)$, denote by $B^i(q)$ the budget set faced by agent i at prices q ; i.e., the set of sequences (c, θ) that satisfy conditions (2.1)-(2.2) above, for the vector of security prices q.

2.2. **Definition of equilibrium.** In this section, the strategies available to the agents and the concept of equilibrium used are defined.

A *strategy* for an agent, after a finite history s_t, is simply the choice of a sequence of consumptions $c = (c_{s_t})_{s_t \in S^t, t \in N_+}$ and portfolio $\theta = (\theta_{s_t})_{s_t \in S^t, t \in N_+}$. The strategic market manipulation of agent M, through the awareness of the impact of her trades on market behavior, is described next in our equilibrium concept.

Before defining it, we first make explicit the demand functions of the representative agent for consumption good and assets after every history, as a function of asset prices q. For any vector of assets prices q, let $c^R(q) = \left[c^R_{s_t}(q)\right]_{s_t \in S^t, t \in N_+}$ (resp. $\theta^R(q) = \left[\theta^R_{s_t}(q)\right]_{s_t \in S^t, t \in N_+}$) be the vector of consumption good (resp. assets) demanded by the representative agent. Those vectors are solutions to the problem of maximizing $U^R_0(c)$ subject to $(c, \theta) \in B^R(q)$.

We define next the notion of equilibrium for this setting, which we refer to as *sequential strategic equilibrium*.

Definition 1. A sequential strategic equilibrium is a sequence of prices $\widetilde{q}$ and a sequence of consumptions $(\widetilde{c}^M, \widetilde{\theta}^M)$ for the strategic agent such that, taking as given the demand functions (c^R, θ^R) and prices $\widetilde{q}$, the sequence $(\widetilde{c}^M, \widetilde{\theta}^M)$ maximizes the expression $U^M_0(c)$ subject to

$$(c, \theta) \in B^M(q), \tag{2.3}$$

$$c_{s_t} + c^R_{s_t}(q) = w_{s_t} \text{ and} \tag{2.4}$$

$$\theta_{s_t} + \theta^R_{s_t}(q) = \underbrace{(0, ..., 0)}_{L \text{ times}} \text{ for every } s_t. \tag{2.5}$$

Constraints (2.4) and (2.5) above capture the market manipulation. They specify the price formation through market clearing conditions, and are known to the strategic agent. Thus, fully aware of the direct impact of her consumptions and asset holdings on asset prices, the strategic agent directly manipulates the market to her best interest. Moreover, in equilibrium, the price-takers behave as usual competitive agents, unaware of the market manipulation.

One can notice, in the above definition, that constraint (2.3) in the strategic agent program is satisfied as long as the constraints (2.4) and (2.5) are satisfied. The constraint (2.3) is left in the program to make explicit that the consumption of the monopolist must be affordable to her. The redundancy comes as a consequence of the definition, and not as part of the definition.

The above definition is close in its spirit to the definition of *Nash Competitive Equilibrium* as introduced in Hens et al. [8]. Our concept simply extends it to an intertemporal setting.

We can also notice that every equilibrium vector of prices is *arbitrage-free*, since otherwise the representative agent would have an infinite demand for some asset after at least one history and Constraint (2.5) would be violated. It is then shown in Huang and Werner [9] that, when a vector of prices q is arbitrage-free, there exists a sequence of strictly positive numbers $\{\pi_{s_t}\}_{s_t \in S^t, t \in N_+}$ with $\pi_{s_0} = 1$ such that:

$$\pi_{s_t} q^j_{s_t} = \pi_{s_{t+1}},$$

where $s_{t+1} = (s_t, j)$, for every j $(j = 1, ..., J)$ and s_t $(s_t \in S^t$ and $t \in N_+)$. The sequence of prices π is often called the *event prices* associated with q; it obtains independently of the market structure from the no-arbitrage condition.

We also need to rule out the possibility of rolling over any debt through excessive future borrowing, also known as Ponzi's scheme. To avoid this issue, we assume that every agent cannot borrow more than the present value of her current endowment. Formally, we assume that for any vector of prices q, every portfolio strategy satisfies the *wealth constraint*:

$$q_{s_t}\theta_{s_t} \geq -\frac{1}{\pi_{s_t}} \sum_{s_\tau \in C(s_t)} \pi_{s_\tau} w^i_{s_\tau},$$

for every s_t ($s_t \in S^t$ and $t \in N_+$).

A sequential strategic equilibrium with the property that the set of feasible portfolio strategy of each agent satisfies the wealth constraint is called a *sequential strategic equilibrium under the wealth constraint (SSEWC).* Further assumptions are needed to prove existence of an SSEWC; those assumptions are presented in Section 4.

3. Survival and accuracy of beliefs

In this section, we define the concepts of vanishing and surviving along a path for an agent, a measurement of accuracy of belief and a notion of entropy of beliefs related to it. Most of these definitions can be found in Sandroni [15] or Lehrer and Smorodinsky [11].

3.1. **Long-run consumption.** Throughout this paper, We focus on the long-run evolution of the consumption of an agent. Sandroni in [15] makes his analysis based on accumulation of wealth in an economy with sequential markets. We prefer to emphasize a study on long-run consumption since it fits best our consumption-based environment. The below definition is central to the paper.

Definition 2. Agent i ($i = M, R$) vanishes along a path $s \in S^\infty$ if $c^i_{s_t}$ converges to 0 as t converges to infinity. Agent i survives on a path $s \in S^\infty$ if he does not vanish on the path s.

Since the endowment of every agent is uniformly bounded away from 0, an agent can always survive along any path by simply staying in autarchy. It follows naturally that vanishing along a path is a consequence of a bad trading choices.

3.2. **Predictions.** We next define how "close" an agent's belief is from the true probability measure P of the economy, and therefore how good her predictions are.

Definition 3. Agent i ($i = M, R$) eventually makes accurate predictions along a path $s \in S^\infty$ ($s = (s_t, ...)$) if $\left\|P^i_{s_t} - P_{s_t}\right\|_\infty$ converges to 0 as t converges to infinity, where $\|.\|_\infty$ is the standard sup-norm defined on Γ. Agent i merges with the truth if agent i eventually makes accurate predictions $P - a.s.$

In words, an agent makes accurate predictions on a path if the conditional probability of her belief, given any finite history along this path, becomes similar in the long-run to the true conditional probability along this path. We also define a weaker concept of merging with the truth, mainly requiring convergence for a weaker topology.

Definition 4. Agent i ($i = M, R$) eventually makes next period accurate predictions along a path $s \in S^\infty$ ($s = (s_t, ...)$) if $\max\limits_{A \in \Gamma_1} \left|P^i_{s_t}(A) - P_{s_t}(A)\right|$ converges to 0 as t converges to infinity. Agent i weakly merges with the truth if agent i eventually makes next period accurate predictions $P - a.s.$

The concept of next period accurate predictions along a path is close to the concept of accurate predictions, the difference is that subjective conditional probabilities and true conditional probabilities given any finite history become similar next period only. In Lehrer and Smorodinski [11], it is shown that merging implies weak merging, but the converse is not true. Sandroni in [15] gives a counterexample to illustrate this point, and goes into more details in comparing these two concepts. We next define what is meant by inaccurate predictions.

Definition 5. Agent i ($i = M, R$) always makes next period inaccurate predictions along a path $s \in S^\infty$, $s = (s_t, ...)$, if there exists a constant $\eta > 0$ such that $\max\limits_{A \in \Gamma_1} \left|P^i_{s_t}(A) - P_{s_t}(A)\right| > \eta$ for all $t \in N$.

In words, an agent always make inaccurate next period predictions when, along a path, the conditional probability of her belief is uniformly bounded away next period from the true conditional probability.

3.3. **Entropy of beliefs.** In this section we define a notion of "entropy" of beliefs, used later in our analysis. This notion links individual beliefs and intertemporal discount factor. The following notions are already introduced in Lehrer and Smorodinsky [11] and Sandroni [15].

Definition 6. The entropy of agent i's beliefs at period t ($t \in N$) is the Γ_t−measurable function Π^i_t defined by

$$\Pi^i_t = E^P\left(\log\left(\frac{P^i_{t+1}}{P_{t+1}}\right)\middle|\Gamma_t\right).$$

In Lehrer and Smorodinski [11], it is shown that the entropy of agent i's beliefs in period t is always negative, and it is zero if and only if agent i's beliefs and the true probability over the states of nature in the next period are identical. We now turn to the concept of entropy of an agent, as defined in Sandroni [15].

Definition 7. The entropy of an agent i, denoted by Π^i, is the $\Gamma-$measurable function defined by

$$\Pi^i = \log(\beta^i) + \overline{\lim_{t\longrightarrow\infty}} \frac{1}{t} \sum_{1\leq k\leq t} \Pi^i_k.$$

In the above definition, the symbol $\overline{\lim}$ stands for limit sup. This last notion of entropy depends on the intertemporal discount factor of each agent, and also on the limit sup of agents' beliefs.

4. Existence

This section provides a sufficient condition for existence of a SSEWC. To derive this result, we first establish some form of equivalence between our sequential framework and an Arrow-Debreu type of framework, and then we establish existence in this later case.

4.1. **Equivalent model.** In this section, we show that the above framework is equivalent, in a sense defined later, to a new one where contingent decisions are all made in period 0 in an Arrow-Debreu style. This new framework greatly simplifies the analysis, and we will use it throughout. The proof of equivalence follows closely the proof of the equivalence between Arrow-Debreu equilibria and sequential market equilibria in a competitive setting, as described in Wright [17].

Denote by p_{s_t} the price for delivery of the single consumption good in period t after the finite history s_t ($s_t \in S^t$ and $t \in N_+$). The new budget set $F^i(p)$ of agent i ($i = M, R$), given a sequence of prices $p = (p_{s_t})_{s_t\in S^t, t\in N_+}$, is by definition the set of all *positive* consumption streams $(c_{s_t})_{s_t\in S^t, t\in N_+}$ such that:

$$\sum_{s_t\in S^t, t\in N_+} p_{s_t}.c_{s_t} \leq \sum_{s_t\in S^t, t\in N_+} p_{s_t}.w^i_{s_t}.$$

For a sequence of price p, denote by $\overline{c}^R(p) = (\overline{c}^R_{s_t}(p))_{s_t\in S^t, t\in N_+}$ the vector of demand for consumption good for the representative agent; in other words, the vector $\overline{c}^R(p)$ maximizes $U^R_0(c)$ subject to $c \in F^R(p)$. We regard the vector $\overline{c}^R(p)$ as a function of prices p.

A *strategic equilibrium (ME)* is then defined to be a sequence of prices p^M and a sequence of consumption c^M for the strategic agent such that, taking as given the demand functions $\overline{c}^R$ and prices p^M, the sequence c^M maximizes the expression $U_0^M(c)$ subject to

$$c \in F^M(p), \text{ and} \tag{4.1}$$

$$c_{s_t} + \overline{c}^R_{s_t}(p) = w_{s_t} \text{ for every } s_t. \tag{4.2}$$

A strategic equilibrium is similar to sequential equilibrium in that the strategic agent endogenizes the effect on prices of her consumption. The difference is that agents now trade in period 0 contingent claims to deliver consumption goods after every history.

The next result shows that the concept of sequential strategic equilibrium under wealth constraint and the concept of strategic equilibrium are equivalent.

Proposition 1. i) Let$\left(c^M, p\right)$ be a strategic equilibrium. Define $q^j_{s_t} = \frac{p_{(s_t,j)}}{p_{s_t}}$ for every j, every s_t and every t. Then there exits a portfolio strategy θ^M such that (c^M, θ^M, q) is a SSEWC.

ii) Let (c^M, θ^M, q) be a SSEWC. Define $p_{s_t} = \pi_{s_t}$ for every s_t. Then (c^M, p) is a strategic equilibrium.

Proof. See Appendix. □

From now on, we will carry out the analysis with the concept of strategic equilibrium. The above proposition states that it is always possible to come back to the original framework with sequential markets.

It is important to notice that any equilibrium price in a strategic equilibrium must lead to finite wealth for both the strategic agent and the representative agent. Indeed, if otherwise then any of those agents would own an infinite wealth that would result in an infinite demand for the consumption good after any finite history; therefore in this case no equilibrium can exist.

4.2. **Existence.** We now prove existence of a SE, with some restrictions on the demand functions of the market. We first some notations, useful to identify our sufficient condition.

We first notice that the Inada conditions imply that, for every strictly positive prices set by the strategic agent, the demand of the representative after every history is interior.

From the problem of the representative agent in a strategic equilibrium, for every sequence of prices p set by the strategic agent there exists a constant λ_p such that

$$\left(\beta^R\right)^t .dP^R_{s_t}.\left(u^R\right)'(c^R_{s_t}) = \lambda_p.p_{s_t} \tag{4.3}$$

for every s_t (the constant λ_p is the Lagrangian multiplier associated with the program, at prices p).

Denote by Ω the function $\left(u^R\right)'$and by Ω_{s_t} the function $\left(\beta^R\right)^t .dP^R_{s_t}.\left(u^R\right)'$.

It follows that the demand function from the representative agent, after a particular event s_t, is given by

$$c^R_{s_t}(p_{s_t}) = \Omega^{-1}\left(\frac{\lambda_p . p_{s_t}}{\left(\beta^R\right)^t .dP^R_{s_t}}\right).$$

We can notice above that the function Ω is invertible since strictly decreasing by assumption. We need a further assumption to prove existence, presented below.

Assumption 1. *There exists* $X > 0$ such that $\Omega(c).c < X$ *for every* $c \in [0, 2B]$.

Usual utility functions such as the Napierian Logarithm satisfy Assumption 1. This assumption is sufficient to guarantee existence, as shown next.

Theorem 2. Under Assumption 1, there exists a strategic equilibrium.

Proof. See Appendix. □

The proof of existence of a SSEWC follows from the above result and Proposition 1. One can notice that a direct use of dynamic programming tools would not be of any particular help when proving existence of a SSEWC. The reason is that the state-dependant realization of endowment, and the state-dependant demand functions from the market, change the value function of the strategic agent after every finite history. The proof of existence presented above, which uses the equivalence between SSEWC and SE, allows to avoid this issue.

We need another technical assumption on the function Ω above to carry out the remainder of our analysis.

Assumption 2. *There exist* $\varepsilon, \gamma > 0$ *such that for any* $x \in [0, 2B + \gamma]$, *we have*

$$-\frac{\Omega(x)}{\Omega'(x)} > x - B + \varepsilon,$$

where B *is the upper-bound on the endowment process.*

This assumption is satisfied by a large class of utility function. For instance, if we assume that the one-period utility function of the price-taking agents is the Napierian logarithm or any function $u(x) = x^\rho$ for some $\rho \in (0, 1)$ (only cases to get strict concavity), then the utility function will also satisfy both Assumptions 1 and 2. From this point on, Assumptions 1-2 are assumed.

5. Long-run performance of the strategic agent

In this section, we study the equilibrium long-run behavior of the strategic agent. In particular, we show that depending on her subjective beliefs the strategic agent may very well vanishes in the long-run.

The next result presents a case where the strategic agent vanishes. In this case, the strategic agent faces a market making accurate predictions.

Theorem 3. Assume that $\beta^R = \beta^M$, and that the representative agent eventually makes accurate predictions. In every equilibrium, and P-a.s.,

a) If the strategic agent does not eventually make accurate predictions on a path $s \in S^\infty$ then he vanishes on this path.

b) If the strategic agent eventually makes accurate predictions on a path $s \in S^\infty$ then he survives on this path.

Proof. See Appendix. □

Theorem 3 is important in order to understand how essential a good knowledge of the uncertainty is. It states that $P-a.s$, even the strategic agent must make accurate predictions in order to survive, in the case where the representative agent forecasts accurately and has the same discount factor as the strategic agent. Sandroni gets a similar result for price-taking agents only, so this result seems to be robust to perturbations of a given model.

Also, a direct consequence of Theorem 3 can be stated as follows: the strategic agent survives P^M-a.s; that is, the strategic agent believes that he will survive.

The intuition of Theorem 3 is central to the paper: the strategic agent chooses to make her trading plans to increase her consumption along the paths that he believes to be the most likely to occur. If the market is willing to increase its consumption over different paths than the strategic agent choices, then the strategic agent will give up consumption along the paths privileged by the market to increase even more her consumption on her privileged paths.

If the paths the strategic agent neglects are actually more likely to occur, then the strategic agent ultimately vanishes.

We consider next the case where no particular assumptions on the intertemporal discount factor are placed. Instead, we compare the entropy of the strategic agent to the entropy of the market to get disappearance of the strategic agent, as in Theorem 3.

We first restate a concept defined in Sandroni [15].

Definition 8. The ratio of beliefs and true probabilities in the next period for an agent i $(i = M, R)$ is bounded if there exist constants v, $V > 0$ such that

$$v < \left| \frac{P_t^i(s)}{P_t(s)} \right| < V \text{ for all } t \in N_+.$$

Under the above restrictions on beliefs, we next show that, when the strategic agent' entropy of beliefs is high enough have high entropy of beliefs.

Theorem 4. Assume that the ratios of beliefs and true probabilities in the next period for agent i ($i = M, R$) are bounded. If $\Pi^M(s) > \Pi^R(s)$ on a path $s \in S^\infty$, then the strategic agent vanishes along the path s.

Proof. See Appendix. □

The above result states that if the representative agent has smaller entropy of beliefs than the strategic agent, then the strategic agent vanishes $P - a.s.$ This result is quite strong, and definitely makes the point of the paper clear: a good knowledge of the uncertainty is essential for the strategic agent to survive.

The previous result completes Theorem 3. Indeed, when an agent makes accurate predictions then her entropy is 0 (and strictly negative otherwise); Theorem 4 follows immediately, provided that the above assumption on beliefs is met.

Also, an immediate corollary of Theorem 4 can be derived: if all the agents have the same beliefs, and if the market is more patient than the strategic agent, then the strategic agent will vanish $P - a.s.$ (This last result follows from Theorem 4 and the definition of entropy of beliefs.)

The next result states under which conditions always making next period inaccurate predictions will drive the strategic agent out of the market. This result extends Proposition 4 in Sandroni [15] to our framework.

Theorem 5. Assume that the ratios of beliefs and true probabilities in the next period for agent i ($i = M, R$) are bounded. Assume also that agent R eventually makes accurate next period predictions on a path, and that $\beta^M = \beta^R$.

Then in every equilibrium, and P-a.s., if the strategic agent always makes inaccurate next period predictions on a path $s \in S^\infty$ then the strategic agent vanishes along the path s.

Proof. See Appendix. □

A point to notice from the previous theorem is that under the above assumptions, when the strategic agent always makes next period inaccurate predictions then he vanishes $P-$a.s.

However, making next period accurate predictions is not a sufficient condition for survival. Indeed, when the strategic agent always makes next period accurate predictions then he can still vanish $P-$a.s, as shown in the following example.

Assume that $S = \{a, b\}$. The true probability of state a is 1 in every period t ($t \in N_+$). Denote by $s \in S^\infty$ the path where state a always occurs. The strategic agent believes that state a occurs in period t ($t \in N_+$) with probability $3^{-\frac{1}{t+1}}$. The price-taker believes that in any period t ($t \in N_+$), state a occurs with probability $2^{-\frac{1}{t+1}}$. Both agents have the same intertemporal discount factor.

Here, the strategic agent eventually makes next period accurate predictions along s, and that the ratios of beliefs and true probabilities over states of nature in the next period for all agents i $(i = M, R)$ are bounded away from zero and infinity. We have that $dP_t^M(s) = \prod_{i=0}^{t-1}\left[3^{-\frac{1}{i+1}}\right]$ and $dP_t^R(s) = \prod_{i=0}^{t-1}\left[2^{-\frac{1}{i+1}}\right]$ for any $t \geq 1$. Therefore for any $t \geq 1$, it is true that

$$\frac{dP_t^M(s)}{dP_t^R(s)} = \prod_{i=0}^{t-1}\left[\left(\frac{3}{2}\right)^{-\frac{1}{i+1}}\right] = \left(\frac{3}{2}\right)^{-\sum_{i=0}^{t-1}\frac{1}{i+1}} \overset{t\longrightarrow\infty}{\longrightarrow} 0,$$

which directly implies that

$$\frac{dP_t^M(s)}{dP_t^R(s)} \overset{t\longrightarrow\infty}{\longrightarrow} 0.$$

Lemma 11, presented later in the Appendix, shows that the strategic agent vanishes along s in this case. However, the path s occurs with true probability 1, and the strategic agent weakly merges with the truth.

6. Appendix

In this section, we provide all the proofs left aside earlier.

6.1. **Proof of Proposition 1.** We now establish the equivalence between SSEWC and SE. The idea of the proof is to show that the agents' budget sets coincide in both concepts, modulo the price transformations specified in the text.

For every c and p such that $c \in F^R(p)$, by the main result in Wright [17], there exists a vector of portfolio θ such that $(c, \theta) \in B^R(q)$ and (c, θ) satisfies the wealth constraint for the representative agent, where $\left(q_{s_t}^j\right) = \frac{p_{(s_t,j)}}{p_{s_t}}$ for every s_t and j. Moreover, fix now any asset prices q, and any $(c, \theta) \in B^R(q)$ satisfying the wealth constraint. Again by the main result in Wright [17], for the system of prices $p = \pi$,where π is associated with q, we have that $c \in F^R(p)$. It follows that $\overline{c}^R(p) = c^R(q)$, where the prices p and q are defined as above, for every p and corresponding q.

Fix now any c and p such that (c, p) satisfies (4.1) and (4.2) above. Define q as previously, and define $\theta_{s_t}^M = -\theta_{s_t}^R(q)$. By construction of $c^R(q)$ and $\theta^R(q)$, and by (4.2), we have that $(c, \theta^M) \in B^M(q)$ and (c, θ^M) satisfies all the other constraints faced by the strategic agent in a SSEWC. Fix any (c, θ, q) satisfying (2.3)-(2.5) and the wealth constraint. For the system of prices $p = \pi$ as previously described, and by construction of $\overline{c}^R(p)$, it must be true that $c \in F^M(p)$.

Thus we have proved that the budget sets of every agent under the two different settings are identical via the above transformations. It follows that the strategic agent choices in both equilibria are the same, modulo the transformation described in the proposition. The proof is now complete.

6.2. **Existence.** We now prove Theorem 2. We first start with a technical lemma, giving an upper-bound of the wealth the strategic agent can accumulate through market manipulation.

Lemma 6. There exists $\gamma > 0$ such that, for every (c, p) satisfying (4.2), we have that

$$\sum_{s_t} p_{s_t}(w^M_{s_t} - c_{s_t}) < \gamma.$$

Proof. We proceed by way of contradiction. Assume that there exist a sequence of real numbers $(\gamma^n)_{n\geq 0}$ converging to infinity, and a sequence (c^n, p^n) satisfying (4.2) such that

$$\sum_{s_t} p^n_{s_t}(w^M_{s_t} - c^n_{s_t}) > \gamma^n. \tag{6.1}$$

In particular, relation (6.1) implies that

$$\sum_{s_t} p^n_{s_t} w^M_{s_t} > \gamma^n, \tag{6.2}$$

and since aggregate endowments are bounded away from 0 and infinity, we also have that

$$\sum_{s_t} p^n_{s_t} w^R_{s_t} > \gamma^n. \tag{6.3}$$

Moreover, since we have that $c^n \in \prod_{s_t\in S^t, t\in N_+} [0, w_{s_t}]$, there exists $\bar{c}$ such that c^n converges to $\bar{c}$ modulo the extraction of a subsequence. By (4.2), the corresponding demands for the representative must satisfy for every s_t

$$c^R_{s_t}(p^n) \to w_{s_t} - \bar{c}_{s_t} < w_{s_t}. \tag{6.4}$$

Since, at the limit, the wealth of the representative agent is infinite by (6.3), this agent must have an infinite demand for some s_t. This contradicts (6.4), and the proof is now complete. □

With the above lemma, we now prove our theorem. Consider the program which consists of maximizing U^M_0 subject to (4.1) and (4.2) at period 0.

Since U^M_0 is strictly increasing, it follows that the constraints (4.1) must hold with equality. By (4.3), any sequence c satisfying constraints (4.1) and (4.2) also satisfies the constraint

$$\sum_{s_t\in S^t, t\in N_+} \Omega_{s_t}(w_{s_t} - c_{s_t}).c_{s_t} = \sum_{s_t\in S^t, t\in N_+} \Omega_{s_t}(w_{s_t} - c_{s_t}).w^M_{s_t}. \tag{6.5}$$

Clearly, a solution to the program consisting of maximizing U^M_0 subject to (6.5) is a strategic equilibrium. We next show that the set of sequences that satisfy (6.5) is non-empty and compact for the product topology.

Non-emptiness: the sequence $(w^M_{s_t})_{s_t\in S^t, t\in N_+}$ satisfies (6.5).

Compactness: For every sequence $c \in \prod_{s_t \in S^t, t \in N_+} [0, w_{s_t}]$, consider the functional

$$\Phi(c) \equiv \sum_{s_t \in S^t, t \in N_+} \Omega_{s_t}(w_{s_t} - c_{s_t}).(w^M_{s_t} - c_{s_t}).$$

By Lemma (6), we can limit the domain of Φ to $D \equiv \Phi^{-1}([o, \gamma])$ in our search for equilibrium.

We next show that Φ restricted on the set D is continuous for the product topology. Fix any arbitrary $\varepsilon > 0$, and consider a sequence of consumptions $(c^n)_{n \in N}$ converging to c for the product topology. We next show that there exists an integer n_0 such that, for every $n \geq n_0$ we have that $|\Phi(c^n) - \Phi(c)| < \varepsilon$, proving continuity.

To show this claim, we break the sum $|\Phi(c^n) - \Phi(c)|$ in two different parts. In a first step, we show that there exists t_0 such that

$$\left| \sum_{s_t \in S^{t+t_0}, t \in N_+} \left[\Omega_{s_t}(w_{s_t} - c^n_{s_t}).(w^M_{s_t} - c^n_{s_t}) - \Omega_{s_t}(w_{s_t} - c_{s_t}).(w^M_{s_t} - c_{s_t}) \right] \right| < \frac{\varepsilon}{2}$$

for every n. Then, we use the continuity of Ω_{s_t}, for the first $\sum_{t=0}^{t_0} L^t$ events s_t,[10] to show that the difference $|\Phi(c^n) - \Phi(c)|$ restricted to the first $\sum_{t=0}^{t_0} L^t$ events can be made arbitrarily small for n large enough.

Fix first any arbitrary $x, y \in \prod_{s_t \in S^t, t \in N_+} [0, w_{s_t}]$, and any period p. We have that

$$\begin{aligned}
& \left| \sum_{s_t \in S^{t+p}, t \in N_+} \begin{bmatrix} \Omega_{s_t}(w_{s_t} - x_{s_t}).(w^M_{s_t} - x_{s_t}) \\ -\Omega_{s_t}(w_{s_t} - y_{s_t}).(w^M_{s_t} - y_{s_t}) \end{bmatrix} \right| \\
< & \left| \sum_{s_t \in S^{t+p}, t \in N_+} \begin{bmatrix} \Omega_{s_t}(w_{s_t} - x_{s_t}).(w^M_{s_t} - x_{s_t}) \\ +\Omega_{s_t}(w_{s_t} - y_{s_t}).(w^M_{s_t} - y_{s_t}) \end{bmatrix} \right| \\
< & \left| \sum_{s_t \in S^{t+p}, t \in N_+} \begin{bmatrix} \Omega_{s_t}(w_{s_t} - x_{s_t}).(w_{s_t} - x_{s_t}) \\ +\Omega_{s_t}(w_{s_t} - y_{s_t}).(w_{s_t} - y_{s_t}) \end{bmatrix} \right| \\
\leq & \ 2.X. \sum_{s_t \in S^{t+p}, t \in N_+} \left(\beta^R\right)^t .dP_{s_t}.
\end{aligned}$$

Since $\lim_{p \to \infty} \sum_{s_t \in S^{t+p}, t \in N_+} \left(\beta^R\right)^t .dP_{s_t} = 0$, it follows that there exists a time t_0 such that, for every x and y,

$$\left| \sum_{s_t \in S^{t+t_0}, t \in N_+} \left[\Omega_{s_t}(w_{s_t} - x_{s_t}).(w^M_{s_t} - x_{s_t}) - \Omega_{s_t}(w_{s_t} - y_{s_t}).(w^M_{s_t} - y_{s_t}) \right] \right| < \frac{\varepsilon}{2}.$$

[10]Recall that in any period t, there are L^t possible events, and thus between period 0 and t_0 there are $\sum_{t=0}^{t_0} L^t$ events.

We can apply this result to the sequence $(c^n)_{n\in N}$ and c, to get that for every n

$$\left|\sum_{s_t\in S^{t+t_0},t\in N_+}\left[\Omega_{s_t}(w_{s_t}-c^n_{s_t}).(w^M_{s_t}-c^n_{s_t})-\Omega_{s_t}(w_{s_t}-c_{s_t}).(w^M_{s_t}-c_{s_t})\right]\right|<\frac{\varepsilon}{2}.$$

Moreover, by continuity of the function Ω_{s_t}, we have that, for every history s_t,

$$\Omega_{s_t}(w_{s_t}-c^n_{s_t}).(w^M_{s_t}-c^n_{s_t})\to\Omega_{s_t}(w_{s_t}-c_{s_t}).(w^M_{s_t}-c_{s_t}).$$

Therefore, for every s_t such that $t\le t_0$,there exists $n_{s_t}\in\mathbb{N}$ such that for every $n\ge n_{s_t}$, the following holds:

$$\left|\Omega_{s_t}(w_{s_t}-c^n_{s_t}).(w^M_{s_t}-c^n_{s_t})-\Omega_{s_t}(w_{s_t}-c_{s_t}).(w^M_{s_t}-c_{s_t})\right|<\frac{\varepsilon}{2.\sum\limits_{t=0}^{t_0}L^t}.$$

Define now $n_0=\max\limits_{s_t\in S^t,0\le t\le t_0}n_{s_t}$. For every $n\ge n_0$, we get that:

$$\begin{aligned}
&|\Phi(c^n)-\Phi(c)|\\
=&\left|\begin{array}{l}\sum\limits_{s_t\in S^t,0\le t\le t_0}\left[\Omega_{s_t}(w_{s_t}-c^n_{s_t}).(w^M_{s_t}-c^n_{s_t})-\Omega_{s_t}(w_{s_t}-c_{s_t}).(w^M_{s_t}-c_{s_t})\right]\\
+\sum\limits_{s_t\in S^{t+t_0},t\in N_+}\left[\Omega_{s_t}(w_{s_t}-c^n_{s_t}).(w^M_{s_t}-c^n_{s_t})-\Omega_{s_t}(w_{s_t}-c_{s_t}).(w^M_{s_t}-c_{s_t})\right]\end{array}\right|\\
<&\left|\sum_{s_t\in S^t,0\le t\le t_0}\left[\Omega_{s_t}(w_{s_t}-c^n_{s_t}).(w^M_{s_t}-c^n_{s_t})-\Omega_{s_t}(w_{s_t}-c_{s_t}).(w^M_{s_t}-c_{s_t})\right]\right|\\
&+\left|\sum_{s_t\in S^{t+t_0},t\in N_+}\left[\Omega_{s_t}(w_{s_t}-c^n_{s_t}).(w^M_{s_t}-c^n_{s_t})-\Omega_{s_t}(w_{s_t}-c_{s_t}).(w^M_{s_t}-c_{s_t})\right]\right|\\
<&\sum_{s_t\in S^t,0\le t\le t_0}\left(\frac{\varepsilon}{2.\sum\limits_{t=0}^{t_0}L^t}\right)+\frac{\varepsilon}{2}\\
=&\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

This proves continuity of the function Φ restricted on D for the product topology.

The set of sequences c that satisfy (6.5) is also $\Phi^{-1}(\{0\})$ by construction. Since the functional Φ is continuous on the set D for the product topology, the set $\Phi^{-1}(\{0\})$ is closed in the compact set $\prod\limits_{s_t\in S^t,t\in N_+}[0,w_{s_t}]$, and compactness follows.

Following standard arguments, we have that U^M_0 is also continuous for the product topology on the set $\prod\limits_{s_t\in S^t,t\in N_+}[0,w_{s_t}]$.

Therefore, the program consisting of maximizing U^M_0 subject to (6.5) has a solution, and a strategic equilibrium exists.

6.3. **Proof of Theorem 3.** The following three lemmas are central to all the results in this section, and they represent the keys to understand the whole analysis.

The first lemma is actually Lemma 3 in Sandroni [15]. It is a result in probability theory, independent of the market structure.

Lemma 7. For every agent i ($i = M, R$) and P-a.s, it is true that

$$\infty > \lim_{t \longrightarrow \infty} \frac{dP_t^i}{dP_t} \geq 0.$$

Moreover, agent i ($i = M, R$) eventually makes accurate predictions on a path $s \in S^\infty$ $(P-\text{a.s})$ if and only if

$$\infty > \lim_{t \longrightarrow \infty} \frac{dP_t^i}{dP_t} > 0.$$

The above lemma, as Sandroni points out, states that almost surely an agent i eventually makes accurate predictions on a path s if and only the probability that agent i assigns to s_t does not become arbitrary smaller than the true probability of s_t as t goes to infinity.

The next lemma gives a necessary condition that the equilibrium values must satisfy.

Lemma 8. In any equilibrium, there exists a strictly positive constant $\Theta > 0$ such that, for all $s_t \in S^\infty$ ($t \in N_+$), the following relation holds:

$$\Theta \left(\frac{\beta^M}{\beta^R} \right)^t \frac{dP_t^M}{dP_t^R} u'(c_{s_t}^M) = (w_{s_t}^M - c_{s_t}^M)\Omega'(c_{s_t}^M) + \Omega(c_{s_t}^M) \tag{6.6}$$

By Assumption 2, the right-hand side of (6.6) is necessarily positive and bounded away from 0. The above lemma shows that some restrictions on the utility functions are needed to get interior solutions to the program faced by the strategic agent. However, usual utility functions in economic theory, such as the Napierian logarithm or any function $u(x) = x^\rho$ for any $\rho < 1$, can generate an interior solution satisfying (6.6).

Proof. Consider the problem of maximizing U_0^M subject to (6.5) . This program generates all the equilibrium solutions.

The first-order conditions to the program consisting of maximizing U_0^M subject to (6.5) directly gives equation (6.6), where the constant Θ is the Lagrange multiplier associated with constraint (6.5). This multiplier must also be strictly positive since (6.6) holds with equality. This finishes the proof. □

The next lemma is a direct consequence of Lemma 8.

Lemma 9. In every equilibrium and along any path $s \in S^\infty$,

i) if the market belief is such that

$$\lim_{t\longrightarrow\infty}\left(\frac{\beta^M}{\beta^R}\right)^t\frac{dP_t^M}{dP_t^R}=0$$

then the strategic agent vanishes along the path s.

ii) If there exists $\varepsilon>0$ such that

$$\left(\frac{\beta^M}{\beta^R}\right)^t\frac{dP_t^M}{dP_t^R}>\varepsilon \text{ for all } t\in N_+,$$

then the strategic agent survives on the path s.

Proof. We first prove i) above. Pick any equilibrium and any path $s\in S^\infty$, and assume that the limit of $\left(\frac{\beta^M}{\beta^R}\right)^t\frac{dP_t^M}{dP_t^R}$ is 0.

Consider now (6.6). This equation rewrites as

$$\Theta\left(\frac{\beta^M}{\beta^R}\right)^t\frac{dP_t^M}{dP_t^R}=\frac{(w_{s_t}^M-c_{s_t}^M)\Omega'(c_{s_t}^R)+\Omega(c_{s_t}^R)}{(u^M)'(c_{s_t}^M)}.$$

By Assumption 2, the numerator in the above expression is bounded away from 0. Therefore it must be true that $u'(c_{s_t}^M)$ converges to infinity, and by assumptions on u^M the sequence $(c_{s_t}^M)_{s_t\in S^t,t\in N_+}$ must converge to 0 as t converges to infinity. This proves that the strategic agent vanishes along this path s.

To prove ii), we proceed by way of contradiction. Assume that the strategic agent vanishes along the path s. Then from (6.6) it must true that

$$\Theta\,\varepsilon<\frac{(w_{s_t}^M-c_{s_t}^M)\Omega'(c_{s_t}^R)+\Omega(c_{s_t}^R)}{(u^M)'(c_{s_t}^M)}\ . \tag{6.7}$$

Since the strategic agent vanishes along the path s, it follows that $\left(u^M\right)'(c_{s_t}^M)$ converges to infinity since $c_{s_t}^M$ converges to 0. Therefore, for (6.7) to hold, it must be true that the numerator in its right-hand side converges to infinity.

Also for t large enough it is true that $w_{s_t}^M-c_{s_t}^M>0$, thus $(w_{s_t}^M-c_{s_t}^M)\Omega'(c_{s_t}^R)<0$ by strict concavity of u^R. Thus, it must be true that $\Omega(c_{s_t}^R)$ converges to infinity for (6.7) to hold. It directly follows, by the Inada conditions, that $c_{s_t}^R$ converges to 0, violating the fact that aggregate endowments are bounded away from 0 along the path s. This is a contradiction, and the proof is now complete. □

Intuitively, the above lemma says that, under the assumption that all the agents have the same intertemporal discount factor, if the strategic agent believes that a path is less likely to occur than some other agent does, then the strategic agent will choose to consume less and less in the long-run along this path. Eventually, her optimal hedging plans will make her consume nothing asymptotically along this path. However, if now the strategic agent does not believe that a path is less likely to occur than the price-takers, then he will choose to consume along this path a quantity of good which is uniformly bounded away from 0.

With the three above lemmas, we now prove Theorem 3 as follows.

Consider a set $A \in \Gamma$ as described in Lemma 7. This set is such that $P(A) = 1$ and the properties in Lemma 7 hold for any path in A. Pick any path $s \in A$ such that the representative agent eventually makes accurate predictions on s.

a) In this case, from Lemma 7 we have that $\lim_{t \longrightarrow \infty} \frac{dP_t}{dP_t^R}(s) < \infty$. Since the strategic agent does not make accurate predictions on s, and again from Lemma 7, it follows that $\lim_{t \longrightarrow \infty} \frac{dP_t^M}{dP_t}(s) = 0$. Therefore $\lim_{t \longrightarrow \infty} \frac{dP_t^M}{dP_t^R}(s) = 0$, and by Lemma 9 the strategic agent vanishes along the path s.

b) In this case, we have from Lemma 7 that $\lim_{t \longrightarrow \infty} \frac{dP_t^M}{dP_t}(s) > 0$. Also from Lemma 7, for any agent i $(i = M, R)$ the following relation holds: $\lim_{t \longrightarrow \infty} \frac{dP_t}{dP_t^i}(s) > 0$. It follows that $\lim_{t \longrightarrow \infty} \frac{dP_t^M}{dP_t^R}(s) > 0$, and by Lemma 9 the strategic agent survives on the path s. The proof is now complete.

6.4. **Proof of Theorem 4.** The following lemma is a consequence of the proof of proposition 3 in Sandroni [15], even though Sandroni does not state it as a separate result.

Lemma 10. Assume that the ratios of beliefs and true probabilities in the next period for agent i $(i = M, R)$ are bounded. If the entropy of any agent i $(i = M, R)$ is strictly smaller than the entropy of the other agent, say j, on a path $s \in S^\infty$, then $\lim_{t \longrightarrow \infty} \left(\frac{\beta^i}{\beta^j}\right)^t \frac{dP_t^i}{dP_t^j}(s) = 0$.

Proof. See proof of proposition 3 in Sandroni [15]. The analysis of the above statement does not depend on any market structure. □

Theorem 4 is a direct consequence of the previous lemma, as shown next.

Assume that the ratios described in the theorem are bounded. Also, assume that the entropy of the representative agent is strictly smaller than the entropy of the strategic agent along a path $s \in S^\infty$. It follows from Lemma 10 that $\lim_{t \longrightarrow \infty} \left(\frac{\beta^M}{\beta^R}\right)^t \frac{dP_t^M}{dP_t^R}(s) = 0$. From Lemma 9, the strategic agent vanishes along the path s.

The proof is complete.

6.5. **Proof of Theorem 5.** We first give another lemma in Sandroni [15], making the link between several concepts of accuracy of beliefs presented so far.

Lemma 11. Assume that the ratios of beliefs and true probabilities in the next period for every agent i $(i = M, R)$ are bounded.

Agent i $(i = M, R)$ eventually makes accurate next period predictions on a path $s \in S^\infty$ if and only if $\lim_{t \longrightarrow \infty} \Pi_t^i(s) = 0$.

Agent i $(i = M, R)$ eventually makes inaccurate next period predictions on a path $s \in S^{\infty}$ if and only if there exists a constant $\delta > 0$ such that $\Pi_t^i(s) < -\delta$ for all $t \in N_+$.

The link between entropy and next period accuracy of precision appears from the previous lemma. Next period predictions are accurate if and only the entropy of the beliefs converges to 0, and next period predictions are always inaccurate if and only if the entropy of beliefs of the agent at any point in time are uniformly bounded away from 0.

Also of interest is to notice that, even if the entropy of the strategic agent is not smaller than the entropy of the market, this will not preclude her from vanishing $P-$a.s.

Indeed, consider the following example to illustrate this point. Assume that the intertemporal discount factors of the agents are identical. The price-taker has correct beliefs and the strategic agent weakly merges with the truth, but does not merge with the truth. The above lemma shows that in this case the entropy of the strategic agent is 0, provided that the ratios of her beliefs and true probabilities over states of nature in the next period are bounded away from zero and infinity. The entropy of the price-taker is also 0. Therefore their entropies are identical; nevertheless by Theorem 4 the strategic agent vanishes $P-$a.s.

Theorem 5 follows naturally from the above remarks, as shown next.

Assume that the representative agent eventually makes accurate next period predictions $P-$almost surely, and that the strategic agent always makes inaccurate next period predictions on a path $s \in S^{\infty}$ ($P-$almost surely). Also, assume that the ratios of beliefs and true probabilities in the next period for every agent i $(i = M, R)$ are bounded. It follows by Lemma 11, and the by fact they both have the same intertemporal discount factor, that the entropy of agent i is strictly greater than the entropy of the strategic agent. All the assumptions in Theorem 4 are met, and therefore the strategic agent vanishes $P-$almost surely. This finishes the proof.

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